Uniformly Best Wavenumber Approximations by Spatial Central Difference Operators

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Contents

Philosophy...

Consider a central finite difference stencil

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_j} \approx \sum_{k=1}^{p} c_k^{(p)} (u_{j+k} - u_{j-k})$$

Philosophy

Error in numerical solution is governed by the truncation error of the finite difference stencil.

Strategy

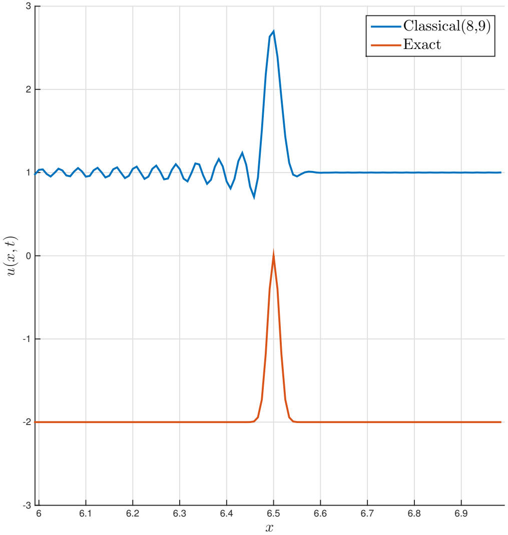
Choose $c_k^{(p)}$ to eliminate terms up to $\mathcal{O}(\Delta x^{2p})$

Example - Advection equation

$$u_t + u_x = 0, \quad 0 \leq x < 6, \quad t \geq 0$$

$$u(x,0) = 2\exp\left(-3200\left(x - \frac{1}{2}\right)^2\right)$$

Periodic Boundary Conditions


Classical(8,9)
Exact
3
2
1
0
-1
-2
-3
$u(x,t)$
6
6.1
6.2
6.3
6.4
6.5
6.6
6.7
6.8
6.9
x

Analytic dispersion relation

$$u_t + u_x = 0$$

Assume u has uniformly convergent Fourier series. Consider general Fourier mode, $u(x,t) = \exp\left(i(\kappa x - \omega t)\right)$

$$\omega = \kappa, \qquad \text{Analytic dispersion relation}$$

$$v_p = \frac{\omega}{\kappa}, \qquad \text{Phase speed}$$

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa}, \qquad \text{Group speed}$$

Speeds independent of wavenumber
$\Rightarrow$ Non-dispersive solution!

Numeric dispersion relation

$$(u_t)_j + \sum_{k=1}^{p} c_k^{(p)} (u_{j+k} - u_{j-k}) = 0$$

$$u_j(t) = \exp\left(i(\kappa x_j - \bar{\omega} t)\right)$$

$$\Rightarrow \underbrace{\bar{\omega}\Delta x}_{\bar{\xi}} = 2\sum_{k=1}^{p} c_k^{(p)} \sin\left(k \underbrace{\kappa \Delta x}_{\xi}\right)$$

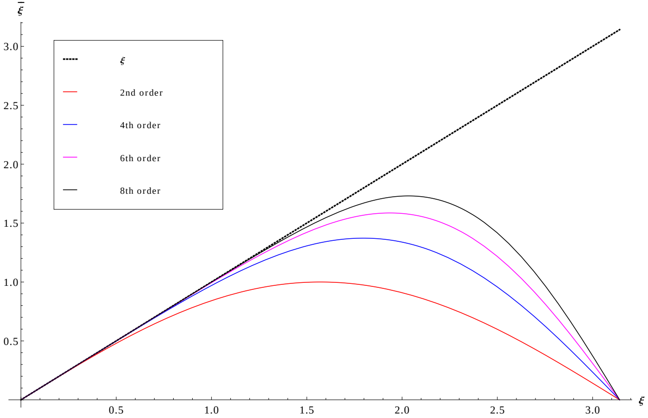
Speeds dependent on wavenumber

$\Rightarrow$ Inherently dispersive solution!

Incorrect phase speed, $\bar{v}_p = \frac{\bar{\omega}}{\kappa}$, and group speed, $\bar{v}_g = \frac{d\bar{\omega}}{d\kappa}$

Can show that $\bar{\xi} < \xi$, $\bar{v}_p < v_p$, $\bar{v}_g < v_g$ for all classical stencils. In fact

$$\xi - \bar{\xi} = \frac{2^p}{\binom{2p}{p}} \int_0^{\xi} (1 - \cos(\xi'))^p d\xi'$$



Alternative philosophy

Problems involving high frequencies / wavenumbers that travel for long times have errors dominated by bad numerical dispersion.

- Fluid dynamics
- Aeroacoustics
- Electromagnetism
- Seismology

Philosophy

Consider dispersion error

$$E(\xi, \mathbf{a}) = \xi - \bar{\xi}(\xi, \mathbf{a})$$

when choosing stencil coefficients

A new problem...

Consider finite difference stencil

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_j} = \sum_{k=1}^{p+n} a_k(u_{j+k} - u_{j-k}) + \mathcal{O}(\Delta x^{2p})$$

$$\bar{\xi} = 2\sum_{k=1}^{p+n} a_k \sin(k\xi)$$

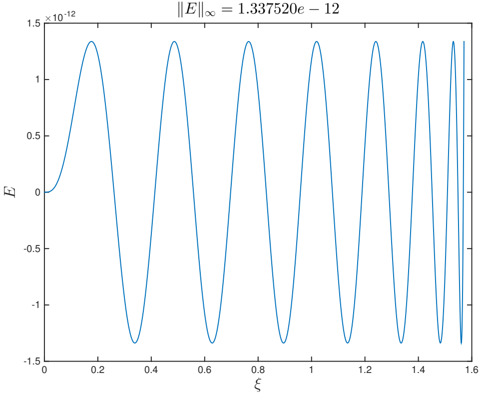
- Accuracy constraint - Order $\mathcal{O}(\Delta x^{2p})$
- Leaves n degrees of freedom
- Use these to minimise dispersion error uniformly, i.e. $\|\xi - \bar{\xi}\|_\infty$

Main results

Theorem

There is a unique stencil $\mathbf{a}$ *that minimises* $\|\xi - \bar{\xi}(\xi, \mathbf{a})\|_\infty$. *The error of this stencil oscillates* $n+1$ *times.*

Can device a convergent algorithm for finding best possible $\mathbf{a}$ (Remez algorithm)



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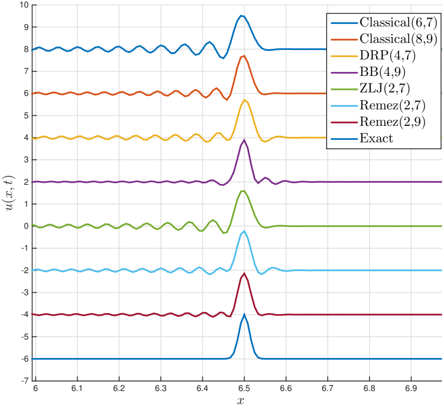

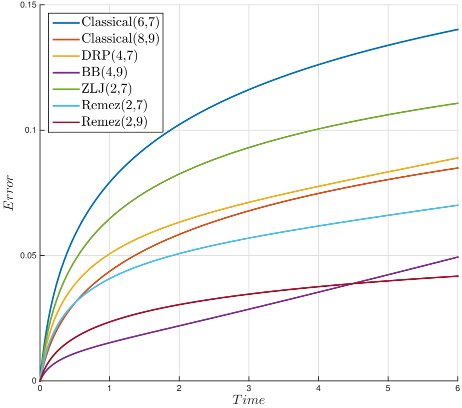

[1] Tam, Webb (1993)
[2] Zingg, Lomax, Jurgens (1996)
[3] Bogey, Bailly (2004)

Conclusion

- For many problems the numerical error comes from inaccurate approximation of Dispersion Relation
- Classical philosophy does not account for this
- We can construct accurate Finite Difference stencils with arbitrarily small Dispersion Error

For more information, see
Linders, Nordström, Journal of Computational Physics, 2015